Announcements

1) Math Career Talks

Today

2:30 CB 1030

Recall: We were proving the following result:

If $f:[a,b]\to\mathbb{R}$ and f is integrable on $[a,b]$ and $\exists\ x_0\in(a,b)$, f is continuous at x_0. If we define $h(x)=\int_a^x f(t)\,dt$, h is differentiable at x_0 and

$$h'(x_0)=f(x_0)$$

Proof:

$$\frac{h(x_0+k)-h(x_0)}{k}$$

$$= \frac{\int_a^{x_0+k} f(t)\,dt - \int_a^{x_0} f(t)\,dt}{k}$$

$$= \frac{\int_{x_0}^{x_0+k} f(t)\,dt}{k}$$

Since f is continuous at x_0, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x_0) - f(y)| < \varepsilon$$

when $|x_0 - y| < \delta$.

Choose $\varepsilon > 0$ and let k be such that $|k| < \delta$.

Then $|(x_0 + k) - x_0| = |k| < \delta$,

so $|f(x_0) - f(y)| < \varepsilon\ \forall$
$y \in (x_0 - k, x_0 + k)$

Then

$$f(x_0)-\varepsilon < f(y) < f(x_0)+\varepsilon,$$

so

$$\frac{\int_{x_0}^{x_0+h}(f(x_0)-\varepsilon)\,dt}{h} \leq \frac{\int_{x_0}^{x_0+h} f(t)\,dt}{h}$$

$$\leq \frac{\int_{x_0}^{x_0+h}(f(x_0)+\varepsilon)\,dt}{h}$$

So then

$$f(x_0)-\varepsilon \le \frac{\int_{x_0}^{x_0+k} f(t)\,dt}{k}$$

$$\le f(x_0)+\varepsilon$$

But $\dfrac{\int_{x_0}^{x_0+k} f(t)\,dt}{k} = \dfrac{h(x_0+k)-h(x_0)}{k}$

So we obtain

$$\left| \frac{h(x_0+k)-h(x_0)}{k} - f(x_0) \right| \le \varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

we conclude

$$\lim_{k \to 0} \frac{h(x_0+k) - h(x_0)}{k} = f(x_0)$$

$$\parallel$$

$$h'(x_0)$$

$\square$

Remark: If $x_0 = a$ or $x_0 = b$, the same proof applies provided you can define f in an open interval about x_0 and maintain integrability.

Integration by Parts

Integrate the product rule.

Given f, g differentiable on $[a,b]$, then if f', g' are integrable,

$$\int_a^b f(x)\,g'(x)\,dx = f(a)g(a) - f(b)g(b) - \int_a^b g(x)\,f'(x)\,dx$$

We know, by integrability, that

$$\int_a^b \left(f(x)g'(x) + g(x)f'(x) \right) dx$$

$$= \int_a^b f(x)g'(x)\,dx + \int_a^b g(x)f'(x)\,dx$$

But with $h(x) = f(x)g(x)$,

$$h'(x) = f'(x)g(x) + g'(x)f(x)$$

and so by the fundamental theorem,

$$\int_a^b \left(f(x)\, g'(x) + g(x)\, f'(x) \right) dx$$

$$= h(b) - h(a)$$

$$= f(b)g(b) - f(a)g(a)$$

and this yields the result.

Example 1: If we interpret

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$$

provided the limit exists,

calculate

$$\int_0^{\infty} x^2 e^{-\frac{x^2}{\alpha^2}}\,dx$$

$$(\alpha > 0)$$

Using multivariate calculus

(later!)

$$\left(\int_0^\infty x^2 e^{-\frac{x^2}{\sigma^2}} \, dx \right)^2$$

$$= \frac{1}{4} \int_0^{2\pi} \int_0^\infty r^5 \cos^2\theta \sin^2\theta \, e^{-\frac{r^2}{\sigma^2}} \, dr \, d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta \int_0^\infty r^5 e^{-\frac{r^2}{\sigma^2}} \, dr$$

$u = \frac{r^2}{\sigma^2}, \; du = \frac{2r}{\sigma^2} dr$

We then get

$$\int_0^\infty r^5 e^{-\frac{r^2}{\alpha^2}} \, dr$$

$$= \frac{\alpha^2}{2} \int_0^\infty u^2 e^{-u} \, du$$

integration by parts

Chapter 6

Sequences of functions

For all $n \in \mathbb{N}$, we want to investigate

$$f_n : [a,b] \to \mathbb{R}$$

with regards to integration.

Definition: Let $f : [a,b] \to \mathbb{R}$.

$(f_n)_{n=1}^{\infty}$ is said to

converge pointwise to f

if $\forall x \in [a,b]$ and $\forall \varepsilon > 0$,

$\exists N = N(x, \varepsilon)$ (N depends on x and ε) such that

$$|f_n(x) - f(x)| < \varepsilon$$

whenever $n \geq N$.

Example 2:

Define $f_n : [0,1] \to \mathbb{R}$

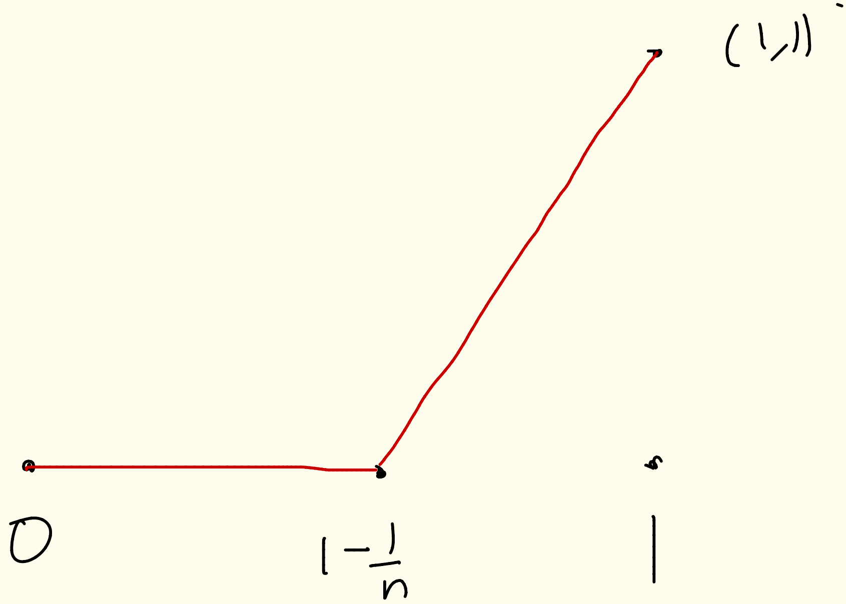
by

$$f_n(x) = \begin{cases} 0, & 0 \le x \le 1 - \frac{1}{n} \\ nx + (1-n), & 1 - \frac{1}{n} \le x \le 1 \end{cases}$$

Claim: $f_n \to f$ where

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

graph of f_n



To prove pointwise convergence

$(f_n \to f)$:

Case 1: $x = 1$

Then $f_n(1) = f(1) = 1 \ \forall n \in \mathbb{N}$

and so the convergence is trivial.

Case 2: $0 \leq x < 1$

Fix $x \in [0, 1)$. Then $\exists$

$N \in \mathbb{N}$, $x < 1 - \frac{1}{N}$.

Then $\forall\ n \geq N$,

$|f_n(x) - f(x)|$

$= |f_n(x)|$

$= 0$ since $x < 1 - \frac{1}{N} \leq 1 - \frac{1}{n}$

$< \varepsilon$ for any $\varepsilon > 0$.

Therefore $f_n \to f$ pointwise on $[0,1]$.

But we see f_n is continuous $\forall x \in [0,1]$ and all $n \in \mathbb{N}$, but f is discontinuous at $x = 1$. How could we fix this and how bad can it get?

The Fix

Definition: (uniform convergence)

If $f_n, f : [a,b] \to \mathbb{R}$ $\forall\, n \in \mathbb{N}$, then $(f_n)_{n=1}^{\infty}$ converges to f uniformly on $[a,b]$ if $\forall\, \varepsilon > 0$ $\exists$ $N \in \mathbb{N}$ such that $\forall\, x \in [a,b]$ and all $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon.$$

Remarks:

1) N depends only on ε, not on x, for uniform convergence

2) If $f_n \to f$ uniformly on $[a,b]$, then $f_n \to f$ pointwise on $[a,b]$ (trivial).

3) If $(f_n)_{n=1}^{\infty}$ is continuous $\forall\, n \in \mathbb{N}$ and $f_n \to f$ uniformly on $[a,b]$, then f is continuous.

4) If $f_n \to f$ uniformly on $[a,b]$ and f_n is integrable $\forall\, n \in \mathbb{N}$ on $[a,b]$, then f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$$

$$= \int_a^b f(x)\,dx$$

5) If f is continuous on $[a,b]$, then $\exists$ a sequence of polynomials $(p_n)_{n=1}^{\infty}$ with

$p_n \to f$ uniformly on $[a,b]$.